Projection of Markov Measures May Be Gibbsian

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We study the induced measure obtained from a 1-step Markov measure, supported by a topological Markov chain, after the mapping of the original alphabet onto another one. We give sufficient conditions for the induced measure to be a Gibbs measure (in the sense of Bowen) when the factor system is again a topological Markov chain. This amounts to constructing, when it does exist, the induced potential and proving its Hölder continuity. This is achieved through a matrix method. We provide examples and counterexamples to illustrate our results.

1. INTRODUCTION

This paper is concerned with the nature of the ''projection'' of a Markov measure, supported by a topological Markov chain (TMC for short), obtained by the action of a factor map mapping the original TMC onto another one. (We recall the definition of a TMC at the beginning of Section 2.) The resulting measure is not expected to be a Markov measure of any order, that is, the resulting process has not a finite memory. The simplest class of measures with infinite memory one could expect is the class of Bowen–Gibbs measures (BGM's for short).

   This problem arises naturally in the coding under restrictions of the kind imposed by forbidding the use of certain blocks. A factor map (called a code in that context) represents a channel with deterministic noise, that is, one which looses information in a predictable way.[16] Here the input

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messages are governed by statistics described by a Markov chain and one wants to determine statistics of output messages.

This problem is also related to the so-called hidden Markov model[19] in Statistics: this model consists in assuming that the observed data are the image of a finite-state Markov chain, this image being obtained by "lumping" some of the states of the state space. Our problem can be phrased by saying that we wish to determine whether a hidden Markov process is distributed according to a BGM.

A third situation where our problem naturally arises is the following. Suppose that a chaotic time series $\{x_n\}_{n \geqslant 0}$ is generated by a deterministic process, a dynamical system, and assume for the sake of definiteness that it is a map on the interval. This means that $x_{n+1} = f(x_n)$. In general one does not have access to f and only observes a symbolic sequence instead of the original orbit. This is because the system can only be observed through a partition of the values of the x_n's that corresponds to the finite precision of the measurement or the computer. A natural question is to determine the invariant measure from this single symbolic sequence, even in the ideal case when the time series would be of infinite length. This problem has been for instance studied in ref. 4 where it was assumed that the observed symbolic sequence is generating by a Bowen–Gibbs measure. A particular class of maps f is the one of piecewise linear Markov maps. When one consider the coding of such maps via the partition given by the intervals corresponding to each branch, the resulting symbolic dynamics is given by a TMC with a state space with k symbols and the invariant measure is a (1-step) Markov measure, k being the number of branches.[9] A basic question is the following: if one observes the dynamical system through a lumping of the partition just introduced, supposing that two atoms of the partition cannot be distinguished, say, then what is the resulting invariant measure describing the time-series ?

A last incarnation of our problem is a one-dimensional lattice gas described by a Markov measure. What happens if, say, two spin values cannot be distinguished ? What we call a projection un the present article is in that context an example of a single site renormalization group transformation. Non Gibbsianess is not expected since there are no phase transitions in one dimensional finite range systems. Some useful references for the reader interested in classical models of statistical mechanics are refs. 5, 12, 14, and 15. Of course, while we restrict ourselves to the one-dimensional setting, the problem of transforming Gibbs measures (by many other types of procedures) can be set in the much more general context of measures on d-dimensional lattices, see ref. 6 for the most recent review.

Description of the paper. Section 2 is devoted to the set-up of our article. We also give the ansatz for the induced potential based on a simple

property of a Gibbs measure. It turns out that the point is to control an infinite product of non-square matrices.

In Section 3 we state our main result, namely some sufficient conditions to get a BGM from the original 1-step Markov measure after the projection of its state space. We emphasize that the presence of forbidden blocks in the original system causes the main difficulty. The projection process induces some strong topological correlations in the resulting system and the existence of the ansatz potential is not obvious at all.

The main result is proved in Section 4. In a first subsection we define a suitable projective metric which is the central tool to control the infinite products of matrices appearing in the ansatz of the induced potential. In the following subsection we state a theorem giving some sufficient conditions on a point in the projected TMC in order to have a well-defined potential at that point. In the last subsection we show how to extend the preceding theorem to the whole projected TMC and we prove the Hölder continuity of the induced potential. Therefore, under suitable conditions, the projection of the initial Markov measure is a BGM.

Section 5 provides a typical example illustrating our main result. Then we also consider the case when the original TMC is a full shift, that is when no blocks are forbidden. It turns out that the projected measure is always a BGM, generically with an infinite range potential (in very special cases the potential can be of finite range). Notice that in the absence of forbidden blocks our problem is considerably simplified. We also present an example showing that one of the two hypothesis needed to establish our main result is not just technical. Indeed in that example the induced potential is not defined at some point (the infinite product mentioned above does not converge). This also illustrates the non-trivial effect that the presence of forbidden blocks in the original system may have. Then we give a formula of the induced potential, when it is well-defined, at periodic points. This follows from the classical Perron–Frobenius theorem since we have for such points to perform products of positive square matrices.

In Section 6 we give some links between our paper and some related works both in our context and in other settings. We also address some natural issues raised by our main result and the counterexamples.

A last section contains the proof of some auxiliary lemmas.

2. SET-UP AND ANSATZ FOR THE INDUCED POTENTIAL

2.1. Set-Up

Let (A_M, σ) be a TMC where A stands for the (finite) alphabet, M for the incidence matrix and σ denotes the shift map. This means that M is

a $0-1$-matrix selecting a subset of all possible infinite sequences $\mathbf{a} = \mathbf{a}(0)\,\mathbf{a}(1)\cdots$ drawn from the alphabet A:

$$A_M := \{\mathbf{a} \in A^{\mathbb{N}}: M(\mathbf{a}(i), \mathbf{a}(i+1)) = 1 \ \forall i \in \mathbb{N}\}\,.$$

This subset is closed under the action of the shift transformation σ, that is, $\sigma A_M = A_M$, where σ is defined as follows: $(\sigma\mathbf{a})(i) := \mathbf{a}(i+1)$ for any $\mathbf{a} \in A_M$. (TMC's are nothing but subshifts of finite type with forbidden block of length two.) Let us recall that a TMC can be viewed as the set of infinite paths on the directed graph (digraph) with vertex set equal to the alphabet and arrows corresponding to allowed transitions between symbols of the alphabet according to the incidence matrix.[10] (We will use this representation in Section 5.)

Suppose (A_M, σ) is *topologically mixing* or, equivalently, that M is a *primitive matrix*. By this we mean that there is a power $m_0 \geqslant 1$ such that the matrix M^{m_0} has only strictly positive entries. (Notice that this property is equivalent to assume that M is irreducible and aperiodic.[20]) Consider a 1-step Markov measure $\mu: \mathfrak{B}(A_M) \to [0, 1]$, which is σ-invariant and mixing ($\mathfrak{B}(A_M)$ denotes the Borel sigma-algebra of A_M generated by cylinder sets). This measure is a BGM associated to a potential $\phi: A_M \to \mathbb{R}$ which is constant in each cylinder of length two. This potential can be thought as a 2-symbols function, that is a potential of range two. Using the same notation for both the potential and the 2-symbols function, we have $\phi(\mathbf{a}) = \phi(\mathbf{a}(0:1))$, where $\mathbf{a}(0:1)$ denotes the 2-block $\mathbf{a}(0)\,\mathbf{a}(1)$. In general, given $\mathbf{a} \in A_M$ and $0 \leqslant i < j$, the block $\mathbf{a}(i)\,\mathbf{a}(i+1)\cdots\mathbf{a}(j)$ will be denoted by $\mathbf{a}(i:j)$.

The potential generating μ can always be chosen such that

$$\mu[\mathbf{a}(0:n)] = \exp\left(\sum_{i=0}^{n-1} \phi(\mathbf{a}(i:i+1))\right) \mu[\mathbf{a}(n)]\,, \tag{1}$$

where $[\mathbf{a}(0:n)]$ is the cylinder of length $(n+1)$ containing $\mathbf{a}$, i.e., $[\mathbf{a}(0:n)] := \{\mathbf{a}' \in A_M : \mathbf{a}'(i) = \mathbf{a}(i), 0 \leqslant i \leqslant n\}$. (In this case ϕ must be strictly negative.)

Bowen–Gibbs Inequality. Let us recall the following basic characterization of a general BGM. Let Ω be a TMC and $\psi: \Omega \to \mathbb{R}$. Then it is known[2] that there is a unique σ-invariant measure v such that for any $n \in \mathbb{N}_0$ and any admissible $\mathbf{b}$

$$\exp(-K) \leqslant \frac{v[\mathbf{b}(0:n)]}{\exp(\sum_{j=0}^{n} \psi(\sigma^j(\mathbf{b})))} \leqslant \exp(K). \tag{2}$$

where $K > 0$ is a constant independent of n and $\mathbf{b}$. We implicitly put the topological pressure of ψ equal to zero, which is always possible.[18]

From now on, we always choose inside the cohomological class of potentials determining any BGM the normalized one, so, in particular, the one of zero pressure.[18]

Let $\pi: A \to B$ be a map onto another alphabet B. In the sequel we always assume that $\#A > \#B > 1$, that is π is a projection. This map defines a factor system (B_π, σ), with

$$B_\pi := \{\mathbf{b} \in B^{\mathbb{N}} : \exists \mathbf{a} \in A_M \text{ such that } \pi\mathbf{a}(i) = \mathbf{b}(i), i \in \mathbb{N}_0\}. \tag{3}$$

($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We also denote by π the map from A_M onto B_π defined in an obvious way.)

It is readily checked that if B_π is infinite then it has to be uncountable. Notice that (B_π, σ) is in general a topologically mixing sofic subshift.[10] A sofic subshift cannot be described by a list of finite forbidden blocks.

Let us introduce the following distance on B_π

$$d(\mathbf{b}, \mathbf{b}') := \begin{cases} \exp\left(-\dfrac{\min\{j : \mathbf{b}(j) \neq \mathbf{b}'(j)\}}{2(\#B+1)}\right) & \text{if} \quad \mathbf{b} \neq \mathbf{b}' \\ 0 & \text{if} \quad \mathbf{b} = \mathbf{b}'. \end{cases} \tag{4}$$

(The constant dividing the min in this definition is for the sake of later convenience.)

The problem concerning us is to elucidate the nature of the measure $v: \mathfrak{B}(B_\pi) \to [0, 1]$ such that

$$v[\mathbf{b}(0:n)] := \mu\{\mathbf{a} \in A_M : \pi(\mathbf{a}(i)) = \mathbf{b}(i), \forall 0 \leqslant i \leqslant n\}, \tag{5}$$

which is the image (or projection) of the measure μ by π, i.e., $v := \mu \circ \pi^{-1}$.

Some general properties of the original measure are preserved under the action of factor maps. If μ is ergodic so is the measure $v \circ \pi^{-1}$. The same holds for the mixing property. (See, e.g., ref. 9.)

For each $b \in B$ let $E_b := \pi^{-1}(b) \subset A$. For each $bb' \in B \times B$ define the rectangular matrix $\mathcal{M}_{bb'}: E_b \times E_{b'} \to [0, 1]$ by

$$\mathcal{M}_{bb'}(a, a') = \exp[\phi(a, a')] \, M(a, a') \tag{6}$$

for all $aa' \in E_b \times E_{b'}$.

Finally, for each $b \in B$ let us define the column vector

$$\mu_b: E_b \to [0, 1] \quad \text{such that} \quad \mu_b(a) = \mu[a], \quad \text{for any} \quad a \in E_b. \tag{7}$$

A straightforward computation shows that $v: \mathfrak{B}(B_\pi) \to [0, 1]$ satisfies

$$v[b_0 b_1 \cdots b_n] = \mathbb{1}^\dagger \left(\prod_{i=0}^{n-1} \mathcal{M}_{b_i b_{i+1}} \right) \mu_{b_n} \tag{8}$$

for any B_π-admissible cylinder $[b_0 b_1 \cdots b_n]$. Here the symbol $\mathbb{1}$ stands for the all-ones column vector of the adequate dimension (whereas $\mathbb{1}^\dagger$ is the corresponding all-ones row vector).

2.2. Ansatz for the Induced Potential

In the following basic lemmas Ω denotes an arbitrary TMC.

Lemma 1. Suppose that $v: \mathfrak{B}(\Omega) \to [0, 1]$ is a BGM. Then it is associated to the normalized Hölder continuous potential $\psi: \Omega \to \mathbb{R}$ such that

$$\psi(\mathbf{b}) = \lim_{n \to \infty} \log \left(\frac{v[\mathbf{b}(0:n)]}{v[\mathbf{b}(1:n)]} \right) \tag{9}$$

for any $\mathbf{b} \in \Omega$.

We refer to ref. 18 for the straightforward proof of this result. We remark that $\psi_n(\mathbf{b}) := \log(v[\mathbf{b}(0:n)]/v[\mathbf{b}(1:n)])$ defines for each $n \geqslant 1$ a function which is constant in each cylinder of length $n+1$. This function is the Hölder continuous potential of a certain (unique) BGM v_n, say, which is nothing but the n-step Markov approximation of the measure v.

Let us emphasize that Lemma 1 says that *necessarily* the potential of a BGM is given by (9). This is far from sufficient since this lemma holds for any g-measure. A g-measure is an equilibrium state (which is in general not unique) associated to a suitably normalized stictly positive potential which is only continuous. We refer the reader to ref. 17 and references therein for details.

Ansatz for the Induced Potential.   Coming back to our problem, we see that Lemma 1 gives the following ansatz for the induced potential: for each $\mathbf{b} \in B_\pi$, set

$$\begin{aligned}
\psi(\mathbf{b}) &= \lim_{n \to \infty} \log \left(\frac{v[\mathbf{b}(0:n)]}{v[\mathbf{b}(1:n)]} \right) \\
&= \lim_{n \to \infty} \log \left(\frac{\mathbb{1}^\dagger (\prod_{i=0}^{n-1} \mathcal{M}_{\mathbf{b}(i:i+1)}) \mu_{\mathbf{b}(n)}}{\mathbb{1}^\dagger (\prod_{i=1}^{n-1} \mathcal{M}_{\mathbf{b}(i:i+1)}) \mu_{\mathbf{b}(n)}} \right),
\end{aligned} \tag{10}$$

where the second equality is obtained after the straightforward substitutions according to formula (8). We see that this ansatz potential is given by an *infinite product of non square matrices* whose convergence and regularity properties as a function of **b** seem to be not trivial at all. The key tool to control this infinite product will be the use of a suitable projective metric.

In the next section we state (and prove in Section 4) that, under some sufficient conditions, the measure v defined by (5) is the BGM associated to the above ansatz potential. This means that we will show that $\psi(\mathbf{b})$ is correctly defined for every $\mathbf{b} \in B_\pi$ and, moreover, that it is a Hölder continuous function.

3. MAIN RESULT: WHEN A MARKOV MEASURE IS MAPPED TO A BGM

The next theorem gives sufficient conditions ensuring that the function $\mathbf{b} \mapsto \psi(\mathbf{b})$, defined by (10), is well defined and Hölder continuous in the whole projected TMC, B_π. Before stating the theorem, we need some preliminary definitions.

Definition 1 (Row Allowable Matrix). Let E' and E be finite alphabets, and $T: E' \times E \to [0, \infty)$ be a rectangular, non-negative matrix on these alphabets. This matrix is said to be row allowable if for each $e' \in E'$ there exists $e \in E$ such that $T(e', e) > 0$.

This definition is inspired by a very similar one given in ref. 20.

First, we restrict the type of factor maps or projections because we want to get a TMC from the original one. Otherwise, as mentioned above, one would get a sofic subshift in which "topological correlations" are generally "non-local."

Definition 2 (Topological Markov Factor Map). The factor map $\pi: A_M \to B_\pi$ is said to be a topological Markov map if the factor subshift B_π is a TMC.

For each $b \in B$ recall that

$$E_b := \{a \in A : \pi(a) = b\}.$$

For each B_π-admissible block bb', let $M_{bb'} : E_b \times E_{b'} \to \{0, 1\}$ be such that

$$M_{bb'}(a, a') = \begin{cases} 1 & \text{if} \quad M(a, a') = 1, \quad \pi(a) = b \quad \text{and} \quad \pi(a') = b', \\ 0 & \text{otherwise.} \end{cases}$$

We need a further restriction on the factor maps we will be able to handle.

Definition 3 (Full Row Allowable Factor Map). The factor map $\pi: A_M \to B_\pi$ is said to be full row allowable if for each B_π-admissible block bb', the corresponding transition submatrix $M_{bb'}$ is row allowable.

Notice that $M_{bb'}$ is a submatrix of the transition matrix M, which is compatible with the non-negative matrix $\mathcal{M}_{bb'}$ defined by (6). It is clear that the definition of a full row allowable factor map does not depend on the potential ϕ defining the Markov measure on (A_M, σ). It is a purely topological notion.

Denote by $\mathrm{Per}_p(B_\pi)$ the set of admissible periodic points with period $p \geqslant 1$. Notice that there is at least one p with $1 \leqslant p \leqslant \#B$ such that $\mathrm{Per}_p(B_\pi) \neq \varnothing$. This is a basic property of a TMC.[10]

We can now state the main theorem of the paper.

Main Theorem. Suppose that (A_M, σ) is a TMC supporting a 1-step Markov measure μ. Let $\pi: A \to B$ a map from A onto another alphabet B and B_π the corresponding factor space. Assume that $\#A > \#B > 1$ and π satisfies the following conditions:

(H1)   is full row allowable,

(H2)   for each $\mathbf{b} \in \mathrm{Per}_p(B_\pi)$, with $1 \leqslant p \leqslant \#B$, the matrix $\prod_{i=0}^{p-1} \mathcal{M}_{\mathbf{b}(i:i+1)}$ is positive.

Under these hypothesis, the function $\mathbf{b} \mapsto \psi(\mathbf{b})$ defined by (10) is well-defined and Hölder continuous on the whole set B_π which is a TMC. This amounts to saying that the projected measure $\nu = \mu \circ \pi^{-1}$ (remember formula (5)) is the (unique) BGM of the potential ψ, that is, it satisfies the Bowen–Gibbs inequality (2).

In Section 5 we provide typical examples of factor maps satisfying the hypotheses of this theorem. Moreover we give a formula of the induced potential at periodic points. It is natural to ask what happens in the case when A_M is a full shift. We shall show in that section that the projected measure is always a BGM because the hypotheses of our theorem are trivially fulfilled in that case. A more interesting question is whether the corresponding potential can be of finite range (which gives a Markov measure with a certain memory). A typical example will show this is possible but "non generic."

It is worth to point out that the presence of forbidden blocks makes the induced potential of infinite range. The reason is that a potential is of

finite range if and only if the sequence in formula (9) becomes constant after some n_0, which means that the potential is of range $n_0 + 1$ (see the lines just after Lemma 1). The presence of forbidden blocks makes unlikely this phenomenon to occur (see formula (10)).

An example built in Section 5 will show that the hypothesis H2 is unavoidable since we will exhibit a point such that the induced potential does not exist.

Remark 1. We have only considered the case when the original system is a TMC, instead of a more general subshift of finite type (SFT), and the factor map is only a 1-block factor map, instead of, say, a 2-block factor map. A SFT is a subshift for which is given a list of forbidden blocks whose length is two in the case of TMC's. From the mathematical point of view there is no loss of generality since any SFT can be recoded as a TMC and a finite-block factor map as a 1-block factor map. We are not able to handle the case of a generic sofic subshift. We refer to ref. 10 for background informations on symbolic dynamics and coding.

4. PROOF OF THE MAIN THEOREM

This section is divided into three subsections. We first introduce the projective metric we need to control the infinite product of non-square matrices that appears in (10). This is the crucial point in our approach. Then we give sufficient conditions for a $\mathbf{b} \in B_\pi$ such that $\psi(\mathbf{b})$ defined in (10) does exist (Theorem 1). Then we prove our main theorem.

4.1. Contractivity of Positive Non-Square Matrices over Simplices

Let $E \subset A$ be non-empty, and consider the simplex

$$\Delta_E := \left\{ x \colon E \to (0, 1) \colon |x|_1 = 1 \right\}, \tag{11}$$

where $|x|_1 := \mathbb{1}^\dagger x = \sum_{e \in E} x(e)$.

The projective metric in this simplex is the function $\delta_E \colon \Delta_E \times \Delta_E \to [0, \infty)$ such that

$$\delta_E(x, y) := \log \left(\frac{\max_{e \in E} (x(e)/y(e))}{\min_{e \in E} (x(e)/y(e))} \right). \tag{12}$$

The projective metric makes (Δ_E, δ_E) a complete metric space. Of course Δ_E is not complete with respect to the Euclidean, or any other ℓ_p metric.

Let us associate to any matrix $T: E' \times E \to [0, \infty)$ the mapping $F_T: \Delta_E \to \Delta_{E'}$, such that

$$F_T(x) := \frac{Tx}{|Tx|_1}. \tag{13}$$

Notice that this mapping is well defined for any row allowable matrix.

Lemma 2. Let $E, E' \subset A$ be non-empty and $T: E' \times E \to [0, \infty)$ be row allowable, then

$$\delta_{E'}(F_T(x), F_T(y)) \leqslant \delta_E(x, y), \tag{14}$$

for all $x, y \in \Delta_E$.

Proof. We follow the standard procedure one can find in ref. 20. For $y \in \Delta_{E'}$ let $P_y: E' \times E \to [0, 1]$ be such that

$$P_y(e', e) := \frac{T(e', e) \, y(e)}{(Ty)(e')}.$$

for any $(e', e) \in E' \times E$.

Then, for all $e' \in E'$ we have

$$\frac{(F_T(x))(e')}{(F_T(y))(e')} = \frac{|Ty|_1}{|Tx|_1} \times \sum_{e \in E} \left(\frac{P_y(e', e) \, x(e)}{y(e)} \right) = \left(P_y \left(\frac{x}{y} \right) \right)(e'),$$

where $(\frac{x}{y})$ denotes the vector of quotients $\frac{y(e)}{x(e)} \in (0, \infty)^E$.

Since P_y is a stochastic matrix, then

$$\max_{e' \in E'} \frac{(Tx)(e')}{(Ty)(e')} \leqslant \max_{e \in E} \frac{x(e)}{y(e)},$$

$$\min_{e' \in E'} \frac{(Tx)(e')}{(Ty)(e')} \geqslant \min_{e \in E} \frac{x(e)}{y(e)},$$

which implies that $\delta_{E'}(F_T(x), F_T(y)) \leqslant \delta_E(x, y)$. ∎

According to this lemma, a non-negative matrix T defines a "non-expanding" map: $F_T: \Delta_E \to \Delta_{E'}$ for which one defines a "contraction coefficient."

Definition 4. Let $: F_T : \Delta_E \to \Delta_{E'}$ be the mapping defined above (formula (13)). Then this mapping is "contractive" with a "contraction coefficient" equal to

$$\tau(T) := \sup_{x, y \in \Delta_E} \frac{\delta_{E'}(F_T(x), F_T(y))}{\delta_E(x, y)}. \tag{15}$$

According to Lemma 2, this coefficient is never larger than one and under certain conditions it is strictly smaller. In fact one can derive an expression for the contraction coefficient.

Lemma 3 (Contraction Coefficient). For $T : E' \times E \to [0, \infty)$ as above, the contration coefficient (15) is equal to

$$\tau(T) = \frac{1 - \sqrt{\Phi(T)}}{1 + \sqrt{\Phi(T)}}$$

with

$$\Phi(T) = \begin{cases} \min_{e, f \in Ee', f' \in E'} \dfrac{T(e', e)\, T(f', f)}{T(e', f)\, T(f', e)} & \text{if} \quad T > 0 \\ 0 & \text{if} \quad T \geqslant 0. \end{cases}$$

We do not give the proof of this result since it can be deduced straightforwardly *mutatis mutandis* from [ref. 20, Section 3.4] (therein this formula is deduced in the case of square matrices). Let us stress that effective contraction is possible only if the matrix is strictly positive.

4.2. Existence of the Induced Potential at Some Particular Points

The aim of this section is to determine sufficient conditions on **b** under which the limit (10) does exist.

Before investigating the existence of this limit in the whole symbolic space B_π, we shall consider conditions that ensure its existence for particular choices of $\mathbf{b} \in B_\pi$. But before we need to introduce some shorthand notations in order to avoid cumbersome formulas.

Notations. For $\mathbf{b} \in B_\pi$ and for any integers $0 \leqslant m < n$, let

$$\mathscr{M}_{\mathbf{b}(m:n)} := \prod_{i=m}^{n-1} \mathscr{M}_{\mathbf{b}(i:i+1)}. \tag{16}$$

Notice that, since $\mu_{\mathbf{b}(n)} > 0$ (remember (7)) and $\mathcal{M}_{\mathbf{b}(m:n)} \neq 0$, then $\mathcal{M}_{\mathbf{b}(m:n)}\mu_{\mathbf{b}(n)} \neq 0$.

We will assume that $\mathcal{M}_{\mathbf{b}(m:n)}$ is a row allowable matrix (see Definition 1). In this case we have $\mathcal{M}_{\mathbf{b}(m:n)}\mu_{\mathbf{b}(n)} > 0$ and the transformation $F_{\mathcal{M}_{\mathbf{b}(m:n)}}$ defined by the matrix $\mathcal{M}_{\mathbf{b}(m:n)}$ will be simply denoted by $F_{\mathbf{b}(m:n)}$.

Let us write $x_{\mathbf{b}(m:n)} := F_{\mathbf{b}(m:n)}(\hat{\mu}_{\mathbf{b}(n)})$ for the image by $F_{\mathbf{b}(m:n)}$ of the normalized vector $\hat{\mu}_{\mathbf{b}(n)} := \mu_{\mathbf{b}(n)}/|\mu_{\mathbf{b}(n)}|_1$. For $\mathcal{M}_{\mathbf{b}(m:n)}$ row allowable, this vector lies in the simplex $\Delta_{E_{\mathbf{b}(m)}}$ (i. e., $x_{\mathbf{b}(m:n)} > 0$).

Let us denote by $\Delta_{\mathbf{b}(m)}$ the simplex $\Delta_{E_{\mathbf{b}(m)}}$, and by $\delta_{\mathbf{b}(n)}(.,.)$ the corresponding projective distance $\delta_{E_{\mathbf{b}(n)}}(.,.)$.

With the notations just introduced we have

$$\frac{\mathbb{1}^{\dagger}(\prod_{i=0}^{n-1} \mathcal{M}_{\mathbf{b}(i:i+1)})\,\mu_{\mathbf{b}(n)}}{\mathbb{1}^{\dagger}(\prod_{i=1}^{n-1} \mathcal{M}_{\mathbf{b}(i:i+1)})\,\mu_{\mathbf{b}(n)}} = \frac{\mathbb{1}^{\dagger}\mathcal{M}_{\mathbf{b}(0:n)}\mu_{\mathbf{b}(n)}}{\mathbb{1}^{\dagger}\mathcal{M}_{\mathbf{b}(1:n)}\mu_{\mathbf{b}(n)}}$$
$$= \mathbb{1}^{\dagger}\mathcal{M}_{\mathbf{b}(0:1)}F_{\mathbf{b}(1:n)}(\hat{\mu}_{\mathbf{b}(n)}) = \mathbb{1}^{\dagger}\mathcal{M}_{\mathbf{b}(0:1)}x_{\mathbf{b}(1:n)}.$$

Therefore, proving that limit (10) exists amounts to proving that

$$\psi(\mathbf{b}) := \lim_{n \to \infty} \log(\mathbb{1}^{\dagger}\mathcal{M}_{\mathbf{b}(0:1)}x_{\mathbf{b}(1:n)}) \tag{17}$$

exists. In fact, under suitable conditions, $x_{\mathbf{b}(1:n)}$ converges exponentially fast to a certain vector, as $n \to \infty$.

Theorem 1. Let $\mathbf{b} \in B_\pi$ be such that:

(H1') for each $i \geq 0$, $\mathcal{M}_{\mathbf{b}(i:i+1)} : E_{\mathbf{b}(i)} \times E_{\mathbf{b}(i+1)} \to (0, \infty)$ is row allowable;

(H2') there exists a bounded gap, strictly increasing sequence $\ell(0) < \ell(1) < \cdots$ (i.e., $0 < \ell(k+1) - \ell(k) < s$ for all $k \in \mathbb{N}$ and some fixed $s \geq 2$), such that $\mathcal{M}_{\mathbf{b}(\ell(k):\ell(k+1))} > 0$ for each $k \in \mathbb{N}_0$.

Under the above conditions, there exists a (normalized) vector $x_{\mathbf{b}(1:\infty)} \in \Delta_{\mathbf{b}(1)}$ and constants $\theta(\mathbf{b}) \in (0, 1)$, $C(\mathbf{b}) > 0$, such that

$$\delta_{\mathbf{b}(1)}(x_{\mathbf{b}(1:\infty)}, x_{\mathbf{b}(1:n)}) \leq C(\mathbf{b})\,\theta(\mathbf{b})^n \quad \forall n \in \mathbb{N}. \tag{18}$$

Moreover, the induced potential ψ at point $\mathbf{b} \in B_\pi$ exists and it satisfies

$$|\psi(\mathbf{b}) - \log(\mathbb{1}^{\dagger}\mathcal{M}_{\mathbf{b}(0:1)}x_{\mathbf{b}(1:n)})| \leq C(\mathbf{b})\,\theta(\mathbf{b})^n \quad \forall n \in \mathbb{N}. \tag{19}$$

Proof of Theorem 1. Hypothesis H1 implies that each one of the matrices $\mathcal{M}_{\mathbf{b}(m:n)}$ is row allowable (a product of row allowable matrices being a row allowable matrix). Therefore, the associated transformation

$F_{\mathbf{b}(m:n)}: \varDelta_{\mathbf{b}(m)} \to \varDelta_{\mathbf{b}(n)}$ is either an isometry or a contraction with respect to the suitable projective metric.

Hypothesis H2 and Lemma 3 imply that $F_{\mathbf{b}(\ell(k):\ell(k+1))}: \varDelta_{\mathbf{b}(\ell(k+1))} \to \varDelta_{\mathbf{b}(\ell(k))}$ is a contraction for each $k \in \mathbb{N}_0$. Let us denote the contraction coefficient of this transformation, $\tau(\mathscr{M}_{\mathbf{b}(\ell(k):\ell(k+1))})$, by $\tau(\mathbf{b}, k)$. (Recall the definition of the contraction coefficient, Definition 4.)

Given $n \in \mathbb{N}$, let $k(n) := \max\{k \in \mathbb{N}_0 : \ell(k+1) \leqslant n\}$. For any $n' > n$ one has

$$\delta_{\mathbf{b}(1)}(x_{\mathbf{b}(1:n)}, x_{\mathbf{b}(1, n')}) = \delta_{\mathbf{b}(1)}(F_{\mathbf{b}(1:n)}(\hat{\mu}_{\mathbf{b}(n)}), F_{\mathbf{b}(1:n)}(x_{\mathbf{b}(1, n')}))$$
$$\leqslant \left(\prod_{j=0}^{k(n)} \tau(\mathbf{b}, j) \right) \times \delta_{\mathbf{b}(n)}(\hat{\mu}_{\mathbf{b}(n)}, x_{\mathbf{b}(n:n')}). \tag{20}$$

(Remember that $\hat{\mu}_{\mathbf{b}(n)} := \mu_{\mathbf{b}(n)}/|\mu_{\mathbf{b}(n)}|_1$.)

The number of different positive matrices $\mathscr{M}_{\mathbf{b}(\ell(k):\ell(k+1))}$, which is at most the number of different blocks $\mathbf{b}(\ell(i):\ell(i+1))$ occurring in $\mathbf{b}$, is finite. Indeed, because of the bounded gap condition, this number is not larger than $(\#B)^{s+1}$, where s is the maximum gap length between consecutive elements in the sequence $\ell(0) < \ell(1) < \cdots$. Thus, $\tau(\mathbf{b}) := \sup\{\tau(\mathbf{b}, k) : k \in \mathbb{N}_0\}$ is a number strictly smaller than 1. Using the definition of $k(n)$ and the bounded gap condition we deduce that $k(n) \geqslant \frac{n-\ell(0)}{s} - 1$. Hence we can write (20) as follows,

$$\delta_{\mathbf{b}(1)}(x_{\mathbf{b}(1:n)}, x_{\mathbf{b}(1, n')}) \leqslant (\tau(\mathbf{b}))^{k(n)} \times \delta_{\mathbf{b}(n)}(\hat{\mu}_{\mathbf{b}(n)}, x_{\mathbf{b}(n:n')})$$
$$\leqslant C_1(\mathbf{b}) \times \theta(\mathbf{b})^n \times \delta_{\mathbf{b}(n)}(\hat{\mu}_{\mathbf{b}(n)}, x_{\mathbf{b}(n:n')}),$$

where $\theta(\mathbf{b}) := \tau(\mathbf{b})^{1/s}$, and $C_1(\mathbf{b}) := 1/\tau(\mathbf{b})^{2+\ell(0)/s}$.

Hence, $x_{\mathbf{b}(1:n)}$ converges exponentially fast to a limit $x_{\mathbf{b}(1:\infty)} \in \varDelta_{\mathbf{b}(1)}$, provided that $\delta_{\mathbf{b}(n)}(\hat{\mu}_{\mathbf{b}(n)}, x_{\mathbf{b}(n:n')})$ is bounded with respect to n and $n' > n$. For this note that

$$\delta_{\mathbf{b}(n)}(\hat{\mu}_{\mathbf{b}(n)}, x_{\mathbf{b}(n:n')}) \leqslant \delta_{\mathbf{b}(n)}(\hat{\mu}_{\mathbf{b}(n)}, x_{\mathbf{b}(n:\ell(k(n)+2))})$$
$$+ \delta_{\mathbf{b}(n)}(x_{\mathbf{b}(n:\ell(k(n)+2))}, x_{\mathbf{b}(n:\ell(k(n)+3))})$$
$$+ \delta_{\mathbf{b}(n)}(x_{\mathbf{b}(n:\ell(k(n)+3))}, x_{\mathbf{b}(n:\ell(k(n)+4))})$$
$$\vdots$$
$$+ \delta_{\mathbf{b}(n)}(x_{\mathbf{b}(n:\ell(k(n)+p))}, x_{\mathbf{b}(n:n')}),$$

with $p = \max\{k \in \mathbb{N}_0 : \ell(k(n)+p) \leqslant n'\}$.

Since each one of the transformations $F_{\mathbf{b}(\ell(k):\ell(k)+1)}$ is contractive with coefficient $\tau(\mathbf{b}, k) \leqslant \tau(\mathbf{b})$, one obtains

$$
\begin{aligned}
\delta_{\mathbf{b}(n)}(\hat{\mu}_{\mathbf{b}(n)}, x_{\mathbf{b}(n:n')}) \leqslant{} & \delta_{\mathbf{b}(n)}(\hat{\mu}_{\mathbf{b}(n)}, x_{\mathbf{b}(n:\ell(k(n)+2))}) \\
& + \tau(\mathbf{b})\, \delta_{\mathbf{b}(\ell(k(n)+2))}(\hat{\mu}_{\mathbf{b}(\ell(k(n)+2))}, x_{\mathbf{b}(\ell(k(n)+2):\ell(k(n)+2))}) \\
& + \tau(\mathbf{b})^2\, \delta_{\mathbf{b}(\ell(k(n)+3))}(\hat{\mu}_{\mathbf{b}(\ell(k(n)+3))}, x_{\mathbf{b}(\ell(k(n)+3):m\ell(k(n)+4))}) \\
& \vdots \\
& + \tau(\mathbf{b})^{p-1}\, \delta_{\mathbf{b}(\ell(k(n)+p))}(\hat{\mu}_{\mathbf{b}(\ell(k(n)+p))}, x_{\mathbf{b}(\ell(k(n)+p):n')}).
\end{aligned}
$$

Finally, since

$$
D := \max\{\delta_{\mathbf{b}_0}(\hat{\mu}_{\mathbf{b}_0}, x_{\mathbf{b}_0\mathbf{b}_1\cdots\mathbf{b}_m}) : 1 \leqslant m < s \text{ and } \mathbf{b}_0\mathbf{b}_1\cdots\mathbf{b}_m \text{ is } B_\pi\text{-admissible}\}
$$

is finite, one gets

$$
\delta_{\mathbf{b}(1)}(x_{\mathbf{b}(1:n)}, x_{\mathbf{b}(1:n')}) \leqslant C(\mathbf{b}) \times \theta(\mathbf{b})^n, \quad \forall n' > n, \tag{21}
$$

where $C(\mathbf{b}) := \frac{D \times C_1(\mathbf{b})}{1-\tau(\mathbf{b})}$. We are done with the proof of Theorem 1.

Now turn to prove (19). By definition of the projective distance and using inequality (21), we obtain

$$
\begin{aligned}
|\underbrace{\log(\mathbb{1}^\dagger \mathscr{M}_{\mathbf{b}(0:1)} x_{\mathbf{b}(1:n')})}_{(*)} - \log(\mathbb{1}^\dagger \mathscr{M}_{\mathbf{b}(0:1)} x_{\mathbf{b}(1:n)})| &\leqslant \delta_{\mathbf{b}(1)}(x_{\mathbf{b}(1:n)}, x_{\mathbf{b}(1,n')}) \\
&\leqslant \left(\frac{D \times C_1(\mathbf{b})}{1-\tau(\mathbf{b})}\right) \times \theta(\mathbf{b})^n \\
&:= C(\mathbf{b}) \times \theta(\mathbf{b})^n.
\end{aligned}
$$

Hence, $\{\log(\mathbb{1}^\dagger \mathscr{M}_{\mathbf{b}(0:1)} x_{\mathbf{b}(1:n)})\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ converging exponentially fast to $\psi(\mathbf{b})$. Formula (19) follows by taking the limit $n' \to \infty$ in the term (*) in the previous chain of inequalities. The proof of Theorem 1 is now finished.

4.3. Proof of the Main Theorem

The proof of the Main theorem consists in obtaining a uniform version of formula (19) in Theorem 1. Before doing this, we need a few more lemmas.

First, we state the following lemma.

Lemma 4. Assume that B_π is a TMC. For each $\mathbf{b} \in B_\pi$ there exists a sequence

$$
0 \leqslant m(0) < \ell(0) < m(1) < \ell(1) < \cdots
$$

such that for each $k \in \mathbb{N}$, $\mathbf{b}(m(k)) = \mathbf{b}(\ell(k))$ and $k(\#B+1) \leqslant \ell(k) < (k+1)(\#B+1)$.

We need also the following lemma which is a partial converse to Lemma 1:

Lemma 5. Let $v \colon \mathfrak{B}(\Omega) \to [0, 1]$ be a σ-invariant measure. Suppose that there is a summable sequence $\{c(n) \in \mathbb{R}^+\}_{n=0}^\infty$, and a Hölder continuous function $\psi \colon \Omega \to \mathbb{R}$ such that

$$\left| \psi(\mathbf{b}) - \log\left(\frac{v[\mathbf{b}(0{:}n)]}{v[\mathbf{b}(1{:}n)]} \right) \right| \leqslant c(n)$$

Then v is the BGM of the (normalized) potential ψ.

(See Section 7 for the proof.)
A last lemma that we need:

Lemma 6. If $\pi \colon A_M \to B_\pi$ is full row allowable, then it is a topological Markov factor map, that is B_π is a (primitive) TMC.

(See Section 7 for the proof.)
It follows from a simple counterexample presented in Section 5, that the converse to Lemma 6 is false.

We are ready for the proof of the main theorem:

For $\mathbf{b} \in B_\pi$, let $m(0) < \ell(0) < m(1) < \ell(1) < \cdots$ be the sequence whose existence is guaranted by Lemma 4. By hypothesis H1 and Lemma 6, we deduce that there is a periodic point inside each one of the cylinders $[\mathbf{b}(m(k){:}\ell(k))]$. Indeed, since the block $\mathbf{b}(m(k){:}\ell(k))$ is B_π-admissible and (B_π, σ) is a TMC, the concatenation $\mathbf{b}(m(k){:}\ell(k)-1)\,\mathbf{b}(m(k){:}\ell(k)-1)\cdots$ is B_π-admissible. Thus, by hypothesis H2 the matrices $\mathcal{M}_{\mathbf{b}(m(k){:}\ell(k))}$ are all positive.

Now by H1 the matrices $\mathcal{M}_{\mathbf{b}(\ell(k-1){:}m(k))}$ are row allowable. It is readily checked that the product of a row allowable matrix by a positive matrix is again a positive matrix, which implies that each one of the matrices $\mathcal{M}_{\mathbf{b}(\ell(k){:}\ell(k+1))}$ is positive.

Therefore, for each $\mathbf{b} \in B_\pi$, assumptions H1' and H2' of Theorem 1 hold, with a gap constant $s = 2(\#B+1)$. Hence, the induced potential exists in the whole space B_π.

Now let us establish the Hölder continuity of ψ.

Since there are finitely many periodic orbits of period less than or equal to $\#B$, then

$$\tau := \sup\{\tau(\mathcal{M}_{\mathbf{b}(0{:}p)}) \colon \mathbf{b} \in \mathrm{Per}_p(B_\pi) \text{ for some } 1 \leqslant p \leqslant \#B\} < 1.$$

Following the proof of Theorem 1, set $\theta := \tau^{1/2(\#B+1)}$,

$$D := \max\{\delta_{b_0}(\hat{\mu}_{b_0}, x_{b_0 b_1 \cdots b_m}) : 1 \leqslant m < 2(\#B+1)\ b_0 b_1 \cdots b_m \text{ is } B_\pi\text{-admissible}\},$$

and $C_1 := 1/\tau^3$.

Inequality (21) holds uniformly in B_π, and we have

$$\delta_{\mathbf{b}(1)}(x_{\mathbf{b}(1:n)}, x_{\mathbf{b}(1:\infty)}) \leqslant \left(\frac{D \times C_1}{1-\tau}\right) \times \theta^n,$$

for all $\mathbf{b} \in B_\pi$ and all $n \in \mathbb{N}$. Hence, formula (19) in Theorem 1 applies uniformly in $\mathbf{b}$, implying that $\psi(\mathbf{b})$ exists for all $\mathbf{b}$ and it is such that

$$\left| \psi(\mathbf{b}) - \log\left(\frac{\nu[\mathbf{b}(0:n)]}{\nu[\mathbf{b}(1:n)]}\right) \right| \leqslant \left(\frac{D \times C_1}{1-\tau}\right) \times \theta^n. \tag{22}$$

On the other hand, from inequality (4.3) it also follows that

$$\delta_{\mathbf{b}(1)}(x_{\mathbf{b}'(1:\infty)}, x_{\mathbf{b}(1:\infty)}) \leqslant \left(\frac{2 \times D \times C_1}{1-\tau}\right) \times \theta^n, \tag{23}$$

for each $\mathbf{b} \in B_\pi$, each $n \in \mathbb{N}$, and all $\mathbf{b}' \in [\mathbf{b}(0:n)]$.

Therefore, because of the definition of the projective distance, we obtain

$$|\psi(\mathbf{b}') - \psi(\mathbf{b})| \leqslant C \times \theta^n,$$

with $C := 2 \times D \times C_1/(1-\theta)$, which implies that

$$\operatorname{var}_n \psi := \sup\{|\psi(\mathbf{b}) - \psi(\mathbf{b}')| : \mathbf{b}_j = \mathbf{b}'_j, 0 \leqslant j \leqslant n\} \leqslant C \times \theta^n. \tag{24}$$

According to the definition of the metric (4), this means the function ψ is Hölder continuous with a Hölder exponent equal to $\log(1/\tau)$, i.e., the logarithm of the inverse of the uniform contraction coefficient. In this way we prove the existence of a Hölder continuous potential, which does satisfy the hypotheses of Lemma 5 because of inequality (22), and the theorem follows.

5. EXAMPLES, COUNTEREXAMPLES AND THE INDUCED POTENTIAL AT PERIODIC POINTS

In this section we provide some examples illustrating the properties of the factor map considered above.

5.1. *Ad Hoc* Example

Define a class of factor maps by the following construction.

1.   Let B_N be a topological Markov chain with transition matrix $N\colon B \times B \to \{0, 1\}$, and let A be a set such that $\#A \geqslant \#B > 1$.

2.   To each $b \in B$ we associate a non-empty subset $E_b \subset A$.

3.   For each two-block bb' admissible in B_N and for each $a \in E_b$, choose a non-empty subset $D_{a,b'} \subset E_{b'}$.

A n-circuit in B_N (corresponding to a circuit in the digraph defined by N) is a B_N-admissible block $b_0 b_1 \cdots b_n$ such that $b_0 = b_n$, and such that $b_i \neq b_j$ for all $0 \leqslant i < j < n$. Note that there is a finite number of n-circuits in B_N for a given n.

The subset $D_{a,b'}$ in item 3 has to be non-empty, but it is arbitrary except for at least one two-block inside each circuit.

We have the following condition.

4.   For each circuit $b_0 b_1 \cdots b_n$ in B_N, choose a two-block $b_i b_{i+1}$. The choice in 3 has to be such that for each $a \in E_{b_i}$, $D_{a, b_{i+1}} := E_{b_{i+1}}$.

5.   Define the transition matrix $M\colon A \times A \to \{0, 1\}$ such that

$$M(a, a') = \begin{cases} 1 & \text{if} \quad a \in E_b, \, a' \in D_{a,b'} \quad \text{and} \quad N(b, b') = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The factor map $\pi\colon A \to B$ such that $\pi^{-1}(b) = E_b$ for each $b \in B$ has all the desired properties.

For the class of examples constructed in this way, any Markov measure in A_M with support in the whole symbolic set, induces a Gibbsian measure in the factor system $B_\pi := B_N$.

One concrete instance of this kind is the topological Markov chain $\{1,\ldots, 5\}_M$ whose digraph is given in Fig. 1.

Together with $\{1,\ldots, 5\}_M$, consider the factor map $\pi\colon \{1,\ldots, 5\} \to \{a, b, c\}$, such that $\pi^{-1}(a) = \{1, 5\}$, $\pi^{-1}(b) = \{2, 4\}$, and $\pi^{-1}(c) = 3$. The factor system $\{a, b, c\}_\pi$ is described by the digraph drawn in Fig. 2.

Notice that one has only two periodic points with period $\leqslant 3$, namely $(ab)^\infty$ and $(ba)^\infty$, and for any Markovian measure with support in $\{1,\ldots, 5\}_M$, the corresponding matrices $\mathscr{M}_{ab}$ and $\mathscr{M}_{ba}$ are both > 0.

5.2. The Induced Potential Is Not of Finite Range Even when the Original TMC Is a Full Shift

The purpose of this section is to see what happens when there are no forbidden blocks in the original system, i.e., it is a full shift. It is obvious
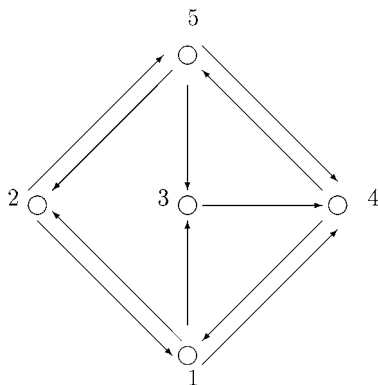
Figure 1.

that the factor system is also a full shift. It will turn out from the example considered hereafter that the image measure of a Markov measure supported by an arbitrary full shift is always a BGM. We are rather interested in the possibility that the induced potential be of finite range. Our example shows that it is indeed possible but somewhat exceptional.

Let μ_ϕ be a Markovian measure for the full shift $\{a, b, c, d\}^{\mathbb{N}}$, and consider the mapping $\pi : \{a, b, c, d\} \to \{0, 1\}$ such that

$$E_0 := \pi^{-1}(0) = \{a, b\} \qquad \text{and} \qquad E_1 := \pi^{-1}(1) = \{c, d\}.$$

This mapping defines a factor map $\pi : \{a, b, c, d\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$.

The induced Gibbs measure has potential $\psi : \{0, 1\}^{\mathbb{N}} \to \mathbb{R}$, which is completely determined by the 2×2 positive matrices $\mathcal{M}_{00}, \mathcal{M}_{01}, \mathcal{M}_{10}, \mathcal{M}_{11}$, and by the 2×1 vectors μ_0 and μ_1.
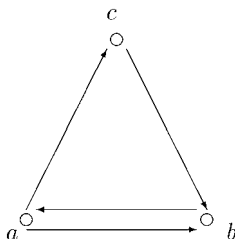

Figure 2.

Indeed, according to the Main theorem (whose hypothesis are trivially satisfied !),

$$\psi(\mathbf{b}) = \lim_{n \to \infty} \log(\mathbb{1}^\dagger \mathcal{M}_{b(0:1)} x_{b(1:n)})$$

$$= \log(\mathbb{1}^\dagger \mathcal{M}_{b(0:1)} x_{b(1:\infty)})$$

$$= \log(\mathbb{1}^\dagger \mathcal{M}_{b(0:1)} F_{b(1:k)}(x_{b(k:\infty)})), \quad \forall k \in \mathbb{N}.$$

Let us remind that for all $1 \leqslant m < n$,

$$x_{b(m:n)} := F_{b(m:n)}(\hat{\mu}_{b(n)}) = F_{b(m:m+1)} \circ F_{b(m+1:m+2)} \circ \cdots \circ F_{b(n-1:n)}(\hat{\mu}_{b(n)}),$$

with $\hat{\mu}_{b(n)} := \mu_{b(n)} / |\mu_{b(n)}|_1$

$$F_{ee'} \colon \varDelta_{e'} \to \varDelta_e, \quad \text{such that} \quad F_{ee'}(x) = \frac{\mathcal{M}_{ee'} x}{|\mathcal{M}_{ee'} x|_1}.$$

In the case we treat now, all these transformations are pure contractions, hence the limit

$$x_{b(k:\infty)} := \lim_{n \to \infty} F_{b(k:n)}(\hat{\mu}_{b(n)})$$

exists for all $k \geqslant 1$. Notice also that the two simplices $\varDelta_0$ and $\varDelta_1$, are equivalent. Because of this, the functions $F_{ee'}$ can be considered as self-maps in the one-dimensional simplex

$$\varDelta := \{x \in (0, 1) \times (0, 1) : x(0) + x(1) = 1\}.$$

For each $e \in \{0, 1\}$, the set of limit points

$$\mathscr{F}_e := \{x_{e(1:\infty)} : \mathbf{e} \in \{0, 1\}^{\mathbb{N}}, \mathbf{e}(1) = e\}, \tag{25}$$

can be thought as a subset of the fractal limit of the Iterated System of Functions $(\varDelta, \{F_{ee'} : ee' \in \{0, 1\} \times \{0, 1\}\})$.

In general $\mathscr{F}_e$ is an uncountable set. For this it is enough that the fixed points of the mappings F_{00} and F_{11} be different.

We are in the situation where the values of ψ can be obtained through linear functionals on $\mathbb{R}^2$, acting on the fractal set $\mathscr{F} \subset \varDelta \subset \mathbb{R}^2$. For each couple $ee' \in \{0, 1\} \times \{0, 1\}$, these functionals are defined by $x \mapsto \mathbb{1}^\dagger \mathcal{M}_{ee'} x$.

Proposition 1. If the induced potential $\psi \colon \{0, 1\}^{\mathbb{N}}$ is of finite range, then at least one of the following conditions must hold.

(1)   The matrices $\mathcal{M}_{00}$ and $\mathcal{M}_{11}$ have the same positive eigenvector.

(2)   One of the matrices $\mathcal{M}_{00}$, $\mathcal{M}_{01}$, $\mathcal{M}_{10}$, or $\mathcal{M}_{11}$, is of rank 1.

(3)   The vector $\mathbb{1}^{\dagger}$ of dimension 1×2 is a left eigenvector for each one of the matrices $\mathcal{M}_{00}, \mathcal{M}_{01}, \mathcal{M}_{10}$, and $\mathcal{M}_{11}$.

Hence, if the Markov measure $\mu_{\phi} \in \mathfrak{B}(\{a, b, c, d\}^{\mathbb{N}})$ is such that, under the factor map $\pi: \{a, b, c, d\} \rightarrow \{0, 1\}$, none of the hypotheses of the previous proposition holds, then the induced Gibbs measure $\nu_{\psi} \in \mathfrak{B}(\{0, 1\}^{\mathbb{N}})$ cannot have a potential of finite range. The space of parameters defining a Markov measure $\mu_{\phi} \in \mathfrak{B}(\{a, b, c, d\}^{\mathbb{N}})$ is the Cartesian product of 4 simplices of dimension 3 (the 4 columns of the probability transition matrix). Conditions (1)–(3) of the previous proposition define a submanifold of dimension not greater that 9 inside that space of parameters: 1 dimension for condition (1), 4 dimensions for condition (2), and 4 dimensions for condition (3). Hence, a Markov measure $\mu_{\phi} \in \mathfrak{B}(\{a, b, c, d\}^{\mathbb{N}})$ generically induces a Gibbs measure whose potential cannot have finite range.

5.3. The Induced Potential at Periodic Points

It is worth to notice that the limit (10) can be effectively computed on periodic points.

Let $\mathbf{b} \in B_{\pi}$ a periodic point of period $p \geqslant 1$. The set

$$\Omega_{\mathbf{b}} := \{\mathbf{a} \in A_{M}: \pi\mathbf{a} = \mathbf{b}\}$$

together with σ^{p}, define the full shift on $E_{\mathbf{b}(0)}$. Indeed, the preimage $\pi^{-1}(\mathbf{b})$ of a periodic point $\mathbf{b} \in \mathrm{Per}_{p}(B_{\pi})$, is a σ^{p}-invariant subset of A_{M}. The system $(\pi^{-1}(\mathbf{b}), \sigma^{p})$ is a TMC whose transition matrix is compatible with $M_{\mathbf{b}(0:p)} := \prod_{i=0}^{p-1} M_{\mathbf{b}(i:i+1)}$. By the hypothesis H2, $M_{\mathbf{b}(0:p)}$ is positive, and hence the system $(\pi^{-1}(\mathbf{b}), \sigma^{p})$ is a full shift on $E_{\mathbf{b}(0)} := \pi^{-1}(\mathbf{b}(0))$.

The topological pressure $P_{\Omega_{\mathbf{b}}}(\phi; \sigma^{p})$ of this system, with respect to the potential ϕ, is the logarithm of the maximal eigenvalue of the matrix $\mathcal{M}_{\mathbf{b}(0:p)}$ (as defined in (16)). Let us denote by $G_{\mathbf{b}(0:p)}$ and $D_{\mathbf{b}(0:p)}$ the left and right eigenvectors associated with $\rho := \exp[P_{\Omega_{\mathbf{b}}}(\phi; \sigma^{p})]$. They are chosen in order that $G_{\mathbf{b}(0:p)}^{\dagger} D_{\mathbf{b}(0:p)} = 1$. Further set $\hat{D}_{\mathbf{b}(0:p)} := D_{\mathbf{b}(0:p)}/|D_{\mathbf{b}(0:p)}|_{1}$.

Proposition 2.   Let $\mathbf{b} \in B_{\pi}$ a periodic point of period $p \geqslant 1$ such that $\mathcal{M}_{\mathbf{b}(0:p)}$ is primitive, which is true, in particular, when hypothesis H2 is satisfied. Then

$$\psi(\mathbf{b}) = P_{\Omega_{\mathbf{b}}}(\phi; \sigma^{p}) - \log(|\mathcal{M}_{\mathbf{b}(1:p)} \hat{D}_{\mathbf{b}(0:p)}|_{1}) . \tag{26}$$

Proposition 2 leads to the following approximation formula. For any $\mathbf{b}' \in B_\pi$ and each $n \in \mathbb{N}$ let $\mathbf{b}$ be a periodic point of minimal period $p(\mathbf{b}, n)$ in $[\mathbf{b}'(0:n)]$. Note that $p(\mathbf{b}, n) \to \infty$ as $n \to \infty$. It follows from (24) that

$$|\psi(\mathbf{b}') - \psi(\mathbf{b})| \leqslant C \times \theta^{p(\mathbf{b}, n)},$$

using for $\psi(\mathbf{b})$ formula (26).

Let us recall that one can get a large class of Gibbs measure as a weak* limit, as $p \to \infty$, of measures concentrated on p-periodic orbits obtained by counting p-periodic orbits weighted by the potential, see, e.g., ref. 9.

Another remarkable property of periodic orbits appears in Livsic's periodic points theorem (see ref. 18): periodic points determine completely the cohomology class of a Hölder continuous potential.

5.4. Example of a Non-Gibbsian Induced Measure

In this section we show that hypothesis H2 is essential to get a well-defined potential on the *whole* factor system. A concrete and simple example is built such that at some point the induced potential is not defined. More precisely, this means the sequence appearing in (17) does not converge. Let us emphasize that this shows the subtle effect produced by the presence of forbidden blocks in the original system. Remember that without forbidden blocks one always gets a potential which is well-defined everywhere (Section 5.2).

Consider the TMC $\{a, b, c, d, e, f\}_M$ defined by the digraph given in Fig. 3.

The mapping $\pi: \{a, b, c, d, e, f\} \to \{0, 1\}$ such that $\pi^{-1}(0) = \{a, b, c, d\}$ and $\pi^{-1}(1) = \{e, f\}$ maps the TMC $\{a, b, c, d, e, f\}_M$ onto the full shift $\{0, 1\}^{\mathbb{N}}$.
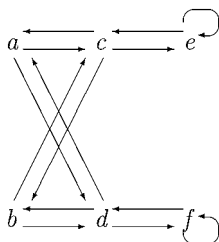

Figure 3.

Supply $\pi: \{a, b, c, d, e, f\} \to \{0, 1\}$ with the with the 1-step Markov measure μ, defined by the probability transition matrix

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 2\gamma & \gamma & 1-3\gamma & 0 \\ 0 & 0 & \gamma & \gamma & 0 & 1-2\gamma \\ 1/4 & 1/4 & 0 & 0 & 1/2 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 1-3\gamma & 0 & 3\gamma-1/2 & 0 \\ 0 & 1/2 & 0 & 1-\gamma & 0 & 2\gamma-1/2 \end{pmatrix},$$

with $1/4 < \gamma < 1/3$. Since $\mathcal{M}$ is double-stochastic, then the one-marginal is the uniform vector $(1/6)\,\mathbb{1}$.

Suppose that the induced measure $\nu = \mu \circ \pi^{-1}$ is a Gibbs measure defined by the potential $\psi: \{0, 1\} \to \mathbb{R}$. If this is the case, one must have $\psi(0^\infty) = \lim_{n \to \infty} \log(\mathbb{1}^\dagger \mathcal{M}_{00} x_{0^\infty(1:n)})$, with

$$\mathcal{M}_{00} = \begin{pmatrix} 0 & 0 & 2\gamma & \gamma \\ 0 & 0 & \gamma & \gamma \\ 1/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 \end{pmatrix} := \begin{pmatrix} \mathbf{0} & A \\ B & \mathbf{0} \end{pmatrix}$$

and $x_{0^\infty(1:n)} = (\mathbb{1}^\dagger \mathcal{M}_{00}^{n-1} \mathbb{1})^{-1} \mathcal{M}_{00}^{n-1} \mathbb{1}$.

Since $\mathcal{M}_{00}$ is a irreducible matrix whose second eigenvalue is zero, the Perron–Frobenius theory gives, for $k \geqslant 3$,

$$\mathcal{M}_{00}^{2k} = \begin{pmatrix} (AB)^k & \mathbf{0} \\ \mathbf{0} & (BA)^k \end{pmatrix} = \rho_{AB}^k \begin{pmatrix} D_{AB} D_{BA}^\dagger & \mathbf{0} \\ \mathbf{0} & D_{BA} D_{AB}^\dagger \end{pmatrix}$$

$$\mathcal{M}_{00}^{2k+1} = \begin{pmatrix} \mathbf{0} & (AB)^k A \\ (BA)^k B & \mathbf{0} \end{pmatrix} = \rho_{AB}^k \begin{pmatrix} \mathbf{0} & D_{AB} D_{BA}^\dagger A \\ D_{BA} D_{AB}^\dagger B & \mathbf{0} \end{pmatrix},$$

where we used the notation previously used for maximal eigenvalues and associated eigenvectors (preceding subsection). Notice that in this case $\rho_{AB} = \rho_{BA} = 5\gamma/4$, $D_{AB} = \frac{1}{5}(32)^\dagger$ and $D_{BA} = (11)^\dagger$. With this we obtain

$$\mathbb{1}^\dagger \mathcal{M}_{00} x_{0(1:2k+1)} = \frac{5\gamma+1}{4}, \qquad \mathbb{1}^\dagger \mathcal{M}_{00} x_{0(1:2k+2)} = \frac{5\gamma}{5\gamma+1}.$$

Since $(5\gamma+1)/4 \neq 5\gamma/(5\gamma+1)$ for $\gamma \neq 1/5$, then ψ is not defined at the fixed point 0^∞.
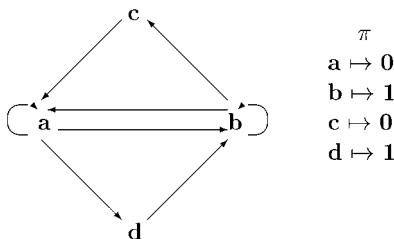
Figure 4.

5.5. The Converse to Lemma 6 Is False

For this consider the topological Markov chain A_M, defined by the digraph which is in Fig. 4.

The factor system defined by A_M and the mapping π defined in Fig. 4 is the full shift $(\{0, 1\}^{\mathbb{N}}, \sigma)$. For this note that the full shift $(\{a, b\}^{\mathbb{N}}, \sigma)$ is a subshift of A_M, and that the factor map restricted to this subshift is a conjugacy.

On the other hand, the submatrix $\mathcal{M}_{01}$ is not a row allowable matrix since there is no symbol $e \in E_1(:= \pi^{-1}(1))$ such that $M(c, e) = 1$. The same is true for M_{10}.

6. CONCLUDING REMARKS AND OPEN QUESTIONS

Some Related Works in Ergodic Theory. Some previous works[3, 11] deal with the study of factor maps between TMC's in the context of ergodic theory and dynamical systems. Let us mention the work of Walters,[22] where the concept of compensation function, which was first considered by Boyle and Tuncel,[3] is used to characterize more general factor maps that those considered in the present work. Walters takes especially advantage of ideas and results from the thermodynamic formalism of equilibrium measures. Let us also quote the recent works[21] where the concept of compensation functions is used to answer certain questions related to measures maximizing some weighted entropy. We point out that in most cases all these works study the behavior of Markov measures under *lifting* while in the present work we were interested in *projecting* Markov measures.

Infinite-To-One Factor Maps and Finite-To-One Ones. There are two classes of factor maps between TMC's. Finite-to-one maps preserve the topological entropy whereas infinite-to-one maps decrease it strictly. (An

infinite-to-one map is a map such that there is at least one point having an uncountable number of preimages.) A simple combinatorial characterization allows to determine whether a factor map is infinite-to-one: one has to check that the map has a diamond. We refer the reader to ref. 10 and Chap. 4 for full details. Hypothesis H2 in the Main theorem (Section 3) implies that π is an infinite-to-one factor map. A necessary condition to have an infinite-to-one factor map is that $\#A > \#B$. But this not sufficient (see the nice example in ref. 10, p. 97).

When the Factor Map Is Not a Topological Markov Map. In general a factor map maps a TMC to a strictly sofic subshift, see ref. 10. BGM's are in fact well-defined on sofic subshifts.[1] We conjecture that ''nice'' factor maps should also map Markov measures to BGM's. We were forced to consider a subclass of topological Markov factor maps, namely full row allowable factor maps (hypothesis H1 in the Main theorem (Section 3)). We believe that this is not a necessary condition but we are not able to prove anything by using our present tools.

About Rational Probability Measures and Semi-Group Measures. In ref. 7, the authors introduce the concept of rational probability measures which are characterized in terms of formal power series. In particular, they show that these measures are exactly the measures obtained by the action of 1-block factor maps (alphabetic monoid morphisms in their context) on 1-step Markov chains. Therefore the Main theorem gives some sufficient conditions for a rational probability measure to be a BGM. The same could be said for semi-group measures that were introduced in ref. 11. Moreover the example of Section 5.4 shows that there are rational probablity measures or semi-group measures that are not Gibbs measures.

Grouped Markov Chains. Our main result generalizes a result by Harris[8] in the context of chains of infinite order (or chains with complete connections). This author calls a grouped Markov chain what we call a projected Markov measure and only considers full shifts, that is, there are no forbidden blocks. Rephrased in his language, we can say that under our hypothesis a grouped Markov shift is continuous with respect to its past with an exponential continuity rate.

Equilibrium Measures with a Non-Hölder Potential. One can relax the hypothesis of Hölder continuity of a potential function and still have a unique equilibrium state satisfying property (2). This is the case when the variation of the potential on cylinders is not exponential (as in the Hölder case) but, for instance, summable. A glance at our proof shows

that the Hölder continuity of the induced potential follows from the exponential convergence in (10), see formula (19). This shows that by using the projective distance we can only obtain a Hölder continuous induced potential.

Weak Gibbs Measures and Hypothesis H2. The example in Section 5.4 provides a simple example of the non-existence of the induced potential at *some point*. This happens because there is the matrix $\mathscr{M}_{00}$ associated to the fixed point 0^∞ which is not strictly positive, in violation with hypothesis H2 of the Main theorem. It may happen that if H2 is satisfied for at least one periodic point the image measure v could be a BGM on a subset of B_π of full measure, that is, the Bowen–Gibbs inequality (2) could hold for v-almost all $\mathbf{b} \in B_\pi$. This situation has been studied, see, e.g., refs. 13 and 23, and such a measure is referred to as a weak Gibbs measure.

7. PROOF OF SOME LEMMAS AND PROPOSITIONS

Proof of Lemma 4. Write $\mathbf{b}$ as the concatenation $\mathbf{b} := \mathbf{b}(0:\#B) \, \mathbf{b}(\#B+1:(\#B+1)+\#B) \cdots$. Inside each one of the factor blocks $\mathbf{b}(k(\#B+1):k(\#B+1)+\#B)$ there is at least one symbol appearing twice, i.e., there are integers $k(\#B+1) \leqslant m(k) < \ell(k) \leqslant k(\#B+1)+\#B$ such that $\mathbf{b}(m(k)) = \mathbf{b}(\ell(k))$. Hence, the lemma is proved.

Proof of Lemma 5. From the hypothesis, for all $\mathbf{b} \in \Omega$ and all $n \in \mathbb{N}_0$ one gets

$$\exp\left(-\sum_{k=0}^{n} c(k) \right) \leqslant \frac{v[\mathbf{b}(0:n)]}{\exp\left(\sum_{j=0}^{n} \psi(\sigma^j(\mathbf{b})) \right)} \leqslant \exp\left(\sum_{k=0}^{n} c(k) \right).$$

Since $\{c(n)\}_{n=0}^{\infty}$ is summable, then the Bowen–Gibbs inequality (2) holds with a constant $K := \sum_{n=0}^{\infty} c(n)$ and we are done.

Proof of Lemma 6. Define the transition matrix $N: B \times B \to \{0, 1\}$ such that

$$N(b, b') = \begin{cases} 1 & \text{if } \exists a, a' M(a, a') = 1, \pi(a) = b \quad \text{and} \quad \pi(a') = b', \\ 0 & \text{otherwise.} \end{cases}$$

We shall prove that the factor subshift B_π and the TMC associated to B_N indeed coincide.

If $\mathbf{b} \in B_\pi$, then there exists $\mathbf{a} \in A_M$ such that $\pi(\mathbf{a}) = \mathbf{b}$. In particular, for each $i \in \mathbb{N}_0$ the block $\mathbf{a}(i, i+1)$ satisfies

$$M(\mathbf{a}(i), \mathbf{a}(i+1)) = 1 \qquad \text{and} \qquad \pi(\mathbf{a}(i)) = \mathbf{b}(i) \forall \quad i \in \mathbb{N}_0.$$

Thus, $N(\mathbf{b}(i), \mathbf{b}(i+1)) = 1$ for each $i \in \mathbb{N}_0$, and hence $\mathbf{b} \in B_N$.

On the other hand, if $\mathbf{b} \in B_N$, then for each $i \in \mathbb{N}_0$ there exists a block $a_i a_i'$ such that

$$M(a_i, a_i') = 1, \quad \pi(a_i) = \mathbf{b}(i) \quad \text{and} \quad \pi(a_i') = \mathbf{b}(i+1).$$

In general there is no reason that $a_i' = a_{i+1}$, but since $M_{\mathbf{b}(i:i+1)}$ is row allowable (recall Definition 1), given a_i and a_i', there exists $a_i'' \in E_{\mathbf{b}(i+2)}$ such that $M(a_i', a_i'') = 1$. Then we can choose $a_{i+1} = a_i'$ and $a_{i+1}' = a_i''$. This choice is such that $a_i a_i' a_i'' \equiv a_i a_{i+1} a_{i+1}'$ is A_M-admissible and $\pi(a_i a_{i+1} a_{i+1}') = \mathbf{b}(i:i+2)$. Thus, starting with $i = 0$, we can proceed by induction in order to obtain a sequence $\mathbf{a} \in A_M$, such that $\pi(\mathbf{a}) = \mathbf{b}$ and $\mathbf{a}(i) = a_i$ for each $i \in \mathbb{N}_0$. Therefore $\mathbf{b} \in B_\pi$. This concludes the proof.

Proof of Proposition 1. Assume that the induced potential $\psi : \{0, 1\}^\mathbb{N} \to \mathbb{R}$ is of range k, for some $k \in \mathbb{N}$. In that case, for all $\mathbf{e} \in \{0, 1\}^\mathbb{N}$,

$$\exp(\psi(\mathbf{e})) = \mathbb{1}^\dagger \mathcal{M}_{(0:1)} F_{\mathbf{e}(1:k)}(x), \quad \forall x \in \mathscr{F}_{\mathbf{e}(k)},$$

with $\mathscr{F}$ as defined in (25).

For this we have the following three logical possibilities: (i) either $\#\mathscr{F}_{\mathbf{e}(k)} = 1$; or (ii) $\#\mathscr{F}_{\mathbf{e}(k)} > 1$, but $F_{\mathbf{e}(1:k)}$ maps all points in $\mathscr{F}_{\mathbf{e}(k)}$ to the same image; or (iii) $\#\mathscr{F}_{\mathbf{e}(k)} > 1$, $F_{\mathbf{e}(1:k)}$ maps two different points in $\mathscr{F}_{\mathbf{e}(k)}$ to two different images, but the linear functional $x \mapsto \mathbb{1}^\dagger \mathcal{M}_{\mathbf{e}(0:1)} x$ maps those different images to the same value.

If (i) holds, then the fixed points of the mappings F_{00} and F_{11}, which belong to $\mathscr{F}_{\mathbf{e}(k)}$, have to coincide. This means that $\mathcal{M}_{00}$ and $\mathcal{M}_{11}$ have the same positive eigenvector, and condition (1) in the statement follows.

If (ii) holds, then $\mathcal{M}_{\mathbf{e}(1:k)}$ is necessarily a rank one matrix, and for this one of the matrices $\mathcal{M}_{00}$, $\mathcal{M}_{01}$, $\mathcal{M}_{10}$, or $\mathcal{M}_{11}$, has to be of rank one. In this way condition (2) in the statement follows.

Finally, if (iii) holds, we need $(\mathbb{1}^\dagger \mathcal{M}_{\mathbf{e}(0:1)})^\dagger$ to be orthogonal to the simplex. In this case we have $\mathcal{M}_{\mathbf{e}(0:1)}^\dagger \mathbb{1} = \alpha \mathbb{1}$. Since $\mathbf{e}(0:1)$ is arbitrary, condition (1) follows. The proof is finished.

Proof of Proposition 2. Since **b** is a periodic point of period p, (10) becomes

$$\psi(\mathbf{b}) = \lim_{n \to \infty} \log \left(\frac{\mathbb{1}^{\dagger} (\mathscr{M}_{\mathbf{b}(0:p)})^{\lfloor \frac{n}{p} \rfloor} \mu_{\mathbf{b}(0:n \bmod p)}}{\mathbb{1}^{\dagger} \mathscr{M}_{\mathbf{b}(1:p)} (\mathscr{M}_{\mathbf{b}(0:p)})^{\lfloor \frac{n}{p} \rfloor - 1} \mu_{\mathbf{b}(0:n \bmod p)}} \right).$$

To ease notation, let, for any **b** and $j \in \mathbb{N}$, $z_{\mathbf{b}(0:j)} := \mathscr{M}_{\mathbf{b}(0:j)} \mu_{\mathbf{b}(j)}$. Now apply Perron–Frobenius theorem[20] to get

$$\psi(\mathbf{b}) = \lim_{n \to \infty} \log \left(\frac{\mathbb{1}^{\dagger} \rho^{\lfloor \frac{n}{p} \rfloor} D_{\mathbf{b}(0:p)} G_{\mathbf{b}(0:p)}^{\dagger} z_{\mathbf{b}(0:n \bmod p)} + O\left(\lambda^{\lfloor \frac{n}{p} \rfloor}\right)}{\mathbb{1}^{\dagger} \mathscr{M}_{\mathbf{b}(1:p)} \rho^{\lfloor \frac{n}{p} \rfloor - 1} D_{\mathbf{b}(0:p)} G_{\mathbf{b}(0:p)}^{\dagger} z_{\mathbf{b}(0:n \bmod p)} + O\left(\lambda^{\lfloor \frac{n}{p} \rfloor - 1}\right)} \right),$$

where λ is any number in $(|\lambda_2|, \rho)$ (λ_2 is the eigenvalue of the next largest modulus after ρ). Therefore

$$\psi(\mathbf{b}) = \log \rho + \lim_{n \to \infty} \log \left(\frac{|D_{\mathbf{b}(0:p)}|_1 + O\left((\lambda/\rho)^{\lfloor \frac{n}{p} \rfloor}\right)}{|\mathscr{M}_{\mathbf{b}(1:p)} D_{\mathbf{b}(0:p)}|_1 + O\left((\lambda/\rho)^{\lfloor \frac{n}{p} \rfloor - 1}\right)} \right),$$

and the proposition follows.

ACKNOWLEDGMENTS

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